

Thermodynamics and cosmological reconstruction in $f(T, B)$ gravity

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Recently, it was formulated a teleparallel theory called $f(T, B)$ gravity which connects both $f(T)$ and $f(R)$ under suitable limits. In this theory, the function in the action is assumed to depend on the torsion scalar T and also on a boundary term related with the divergence of torsion $B = 2\nabla_\mu T^\mu$. In this work, we study different features of a flat FLRW cosmology under this theory. First, we show that the FLRW equations can be transformed to the form of Clausius relation $T_h S_{eff} = -dE + WdV$, where T_h is the horizon temperature and S_{eff} is the entropy which contains contributions both from horizon entropy and an additional entropy term introduced due to the non-equilibrium. We also formulate the constraint for the validity of the generalised second law of thermodynamics (GSLT). Additionally, using a cosmological reconstruction technique, we show that both $f(T, B)$ and $-T + F(B)$ gravity can mimic power-law, de-Sitter and Λ -CMD models. Finally, we formulate the perturbed evolution equations and analyse the stability of some important cosmological solutions.

Keywords: teleparallel gravity; $f(T)$ gravity; $f(R)$ gravity; cosmology; Λ CDM universe ; thermodynamic laws.

I. INTRODUCTION

In current scenario, dark energy (DE) is referred as an active agent which tends to accelerate the expansion in cosmos. The expanding paradigm of the universe has been affirmed from various observational measurements. In 1998, observations of SNeIa accumulated by the high-redshift SN team [1] and SN cosmology project team [2] appeared as illuminating candles in disclosing the expansion of the Universe. The source for this observed cosmic acceleration may be an anonymous energy component entitled as dark energy (DE). In spite of tremendous efforts, late-time cosmic

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acceleration is certainly a major challenge for cosmologists. The direct evidence for cosmic acceleration has strengthened over time with measurements from temperature anisotropies in CMB [3] and BAO [4] which confirm the existence of DE. Dark energy is appeared as enigmatic cosmic ingredient and interpretation of its gravitational effects is a dynamic research field. The most likely theoretical campaigner of DE is the cosmological constant Λ characterized by constant equation of state (EoS) $\omega = -1$ [5]. A number of alternative models have been proposed in this perspective to explain the role of DE in the present cosmic acceleration [6]. The other proposal for the construction of DE models is the modification of Einstein-Hilbert action which lead to modified gravity models. Some important alternative theories of gravity are $f(R)$ gravity [7], $f(R, \mathcal{T})$ gravity (\mathcal{T} is the trace of energy-momentum tensor $\mathcal{T}_{\alpha\beta}$) [11], $f(R, \mathcal{T}, \mathcal{Q})$ gravity (where $\mathcal{Q} = R_{\alpha\beta}T^{\alpha\beta}$) [12, 13], Gauss-Bonnet gravity [14], teleparallel modifications [8, 10, 15], scalar-tensor theories [16, 17], among others.

In current scenarios, generalization of teleparallel theory has gained significant importance, which could provide alternative explanations for the cosmic acceleration [18]. A key problem in $f(T)$ gravity is that it breaks the invariance under local Lorentz transformations, lack of local Lorentz symmetry implies that there is no freedom to fix any of the components of the tetrad [19]. However, in FLRW cosmology, it is always possible to find “good tetrads” to solve this issue [9]. In [10], the authors generalized $f(T)$ by introducing a new Lagrangian $f(T, B)$ which involves a boundary term B which is related to the divergence of the torsion tensor. This theory becomes equivalent to $f(R)$ gravity for the choice of special form $f(-T + B)$. The latter is the only case in which local Lorentz invariance can be achieved.

Cosmological reconstruction in modified theories is one of the significant aspects in cosmology. The reconstruction scheme in $f(R)$ gravity and its modifications have been carried out under different scenarios [20]-[23] to find out realistic cosmology which can explain the transition of matter dominated epoch to DE phase. In this study, one interesting way is to consider the known cosmic evolution and use the field equations to find particular form of Lagrangian that can reproduce the given evolution background. Nojiri et al. [23] executed such reconstruction scheme in order to find some realistic models in $f(R)$ theory which was then applied in $f(R, G)$ and modified Gauss-Bonnet theories [24]. The cosmic evolution based on power law solution of the scale factor has also been discussed in modified theories [25]. Dunsby et al. [26] explored that extra degrees of freedom to the matter component are necessary to reconstruct the Λ cold dark matter (CDM) evolution in $f(R)$ gravity. Carloni et al. [27] set up a new method of reconstructing $f(R)$ gravity using the cosmic parameters rather than any form of the scale factor. In the context of modified

theories, stability of cosmological solutions has been analysed for homogeneous perturbations [28]–[33]. In [29], the stability of $f(R, G)$ models is presented for power law and Λ CDM cosmology. In the context of teleparallel gravity, it was showed that one can reconstruct Λ -CMD universes and describe holographic dark energy models for $f(T)$ gravity [30–32]. As $f(T, B)$ is a generalisation of $f(T)$ and $f(R)$ gravity, it is important to also find out how this theory can reconstruct or mimic different cosmological models. One goal of this work is to reconstruct different cosmological models for $f(T, B)$ gravity and also for the particular choice of the function where $f(T, B) = -T + F(B)$. After that, we will also study the stability of some of these cosmological models.

The connection between the FLRW equations and the first law of thermodynamics (FLT) at the apparent horizon was shown in [34] for $T = 1/2\pi R_A$, $S = \pi R_A^2/G$, where R_A is the radius of the apparent horizon. The Friedmann equations in Gauss-Bonnet gravity and Lovelock gravity were also formulated by using the corresponding entropy formula of static spherically symmetric black holes. Eling et al. [35] found that one cannot find the correct field equations simply by using the Clausius relation in nonlinear theories of gravity. It was remarked that a non-equilibrium treatment of thermodynamics is required, whereby the Clausius relation is modified to $TdS = \delta Q + d_i S$, where $d_i S$ is the entropy production term. Akbar and Cai [36] showed that the Friedmann equations in general relativity (GR) can be written as $dE = TdS + WdV$ (unified FLT on the trapping horizon suggested by Hayward [37, 38]) with the work term being $W = \frac{1}{2}(\rho - p)$. They also extended this work to Gauss-Bonnet gravity [36], Lovelock gravity [36, 39], braneworld gravity [40, 41], $f(R)$ gravity [43] and scalar-tensor gravity [44]. The generalised second law of thermodynamics was also studied in the context of $f(T)$ gravity for different forms of the function [45, 46]. As we have pointed out, the investigation about the validity of thermodynamical laws in modified theories has been carried out by numerous researchers in literature [47]. Here, we are interested to explore the validity of these laws in $f(T, B)$ gravity.

This paper is organised as follows: In Sec. II, we briefly introduce teleparallel equivalent of general relativity and then its generalisation, $f(T, B)$ gravity. Then, we present the basis equations for a FLRW cosmology. Sec. III is devoted to the study of the first and second laws of thermodynamics under this theory. Different reconstructions models are presented in Sec. IV for $f(T, B)$ and also for $-T + F(B)$ gravity. Using perturbation techniques, the stability of different cosmological models are studied in Sec. V. Finally, in Sec. VI we conclude our main results.

II. TELEPARALLEL EQUIVALENT OF GENERAL RELATIVITY AND ITS MODIFICATIONS

Let us briefly introduce the basis of the teleparallel equivalent of general relativity (TEGR). We will use the convention used in Ref. [10] where E_m^μ is the inverse of the tetrad e_μ^m and greek and latin indices refer to space-time and tangent space ones respectively. This theory lies in the idea that the manifold has a vanished curvature but a non-zero torsion. To ensure this kind of geometry, one needs to chose a specific connection where the space is globally flat, the so-called Weitzenböck connection $W_\mu^a{}_\nu$. One important fact it is that this alternative representation of gravity is equivalent (in the field equations) to general relativity. The dynamical variable is the tetrad field and it is related with the metric with the following equation,

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \quad (1)$$

where η_{ab} represents the Minkowski metric $(-, +, +, +)$. Note that the tetrad fields are orthonormal vector at each point of the manifold, hence they obey the following orthogonality relationships

$$E_m^\mu e_\mu^n = \delta_m^n, \quad (2)$$

$$E_m^\nu e_\mu^m = \delta_\mu^\nu. \quad (3)$$

The torsion tensor is constructed by taking the anti-symmetric part of the Weitzenböck connection,

$$T^a{}_{\mu\nu} = W_\mu^a{}_\nu - W_\nu^a{}_\mu = \partial_\mu e_\nu^a - \partial_\nu e_\mu^a. \quad (4)$$

The teleparallel action is then constructed with the torsion scalar which is defined as a contraction of the super potential

$$S^{abc} = \frac{1}{4}(T^{abc} - T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^b - \eta^{ab}T^c), \quad (5)$$

with the torsion tensor $T = S_a{}^{bc}T^a{}_{bc}$. Here, the torsion vector is defined contracting the first two indices of the torsion tensor $T_\mu = T^\nu{}_{\nu\mu}$. Explicitly, the action reads

$$S_{\text{TEGR}} = \frac{1}{\kappa^2} \int e T d^4x + S_m, \quad (6)$$

where e denotes the determinant of the tetrad which is equal to $\sqrt{-g}$ and $\kappa^2 = 8\pi G$. Here, S_m is the action of the matter content. It is possible to prove that the torsion scalar is related with the Ricci scalar directly by

$$R = -T + \frac{2}{e} \partial_\mu (e T^\mu) = -T + B, \quad (7)$$

where B is a boundary term. The Einstein-Hilbert action is constructed with the Ricci scalar R , so that it differs only by a boundary term with respect to the TEGR action. Hence, metric variations of the action (6) are equivalent to tetrad variations of the Einstein-Hilbert action. Therefore, if we vary the action (6) with respect to the tetrad, the corresponding field equations will be identical to the Einstein's field equations.

A well-studied modification of the action (6) is obtained by changing the torsion scalar T to an arbitrary function $f(T)$ which depends smoothly on T . This generalisation then has the following action

$$S_{f(T)} = \frac{1}{\kappa^2} \int e f(T) d^4x + S_m, \quad (8)$$

which gives rise to the $f(T)$ field equations which is a second order theory. This theory is in this sense, analogous to $f(R)$ gravity. However, these two theories are not equivalent. With the aim to combine both $f(R)$ gravity and $f(T)$ gravity, in Ref. [10] it was proposed the following action

$$S_{f(T,B)} = \frac{1}{\kappa^2} \int dx^4 e f(T, B) + S_m, \quad (9)$$

which is a modified teleparallel theory of gravity where now $f(T, B)$ also depends on the boundary term B . In [10] it was proved that by choosing $f = f(T)$ and $f = f(-T + B) = f(R)$ it is possible to recover both $f(T)$ and $f(R)$ gravity respectively. The field equations of this theory are obtained by varying the action with respect to the tetrad giving us,

$$\begin{aligned} 2e\delta_\nu^\lambda \square f_B - 2e\nabla^\lambda \nabla_\nu f_B + eB f_B \delta_\nu^\lambda + 4e \left[(\partial_\mu f_B) + (\partial_\mu f_T) \right] S_\nu^{\mu\lambda} \\ + 4e_\nu^a \partial_\mu (e S_a^{\mu\lambda}) f_T - 4e f_T T^\sigma_{\mu\nu} S_\sigma^{\lambda\mu} - e f \delta_\nu^\lambda = 16\pi e \mathcal{T}_\nu^\lambda, \end{aligned} \quad (10)$$

where $\mathcal{T}_\nu^\lambda = e_\nu^a \mathcal{T}_a^\lambda$ is the standard energy momentum tensor and $\square = \nabla^\mu \nabla_\mu$. In general, this theory is a fourth-order one and it is not invariant under local Lorentz transformations (since T and B are not invariant under local LT). Indeed, the only theory which is invariant under these transformations is obtained by taking $f(T, B) = f(-T + B) = f(R)$, i.e., in the $f(R)$ case.

A. $f(T, B)$ Cosmology

In this section, we will introduce the basis equation of a flat FLRW cosmology in $f(T, B)$ gravity. The metric which describes this space-time in Euclidean coordinates is given by

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2), \quad (11)$$

where $a(t)$ is the scale factor of the universe. In these coordinates, the tetrad field can be expressed as follows

$$e_\mu^a = \text{diag}(1, a(t), a(t), a(t)) . \quad (12)$$

Note that under these coordinates, the issue of non local Lorentz invariance is solved. If we assume that the content of the universe is a perfect fluid and we use the above FLRW tetrad, the $f(T, B)$ cosmology field equations (10) become

$$-3H^2(3f_B + 2f_T) + 3H\dot{f}_B - 3\dot{H}f_B + \frac{1}{2}f(T, B) = \kappa^2\rho_m, \quad (13)$$

$$-3H^2(3f_B + 2f_T) - \dot{H}(3f_B + 2f_T) - 2H\dot{f}_T + \ddot{f}_B + \frac{1}{2}f(T, B) = -\kappa^2 p_m. \quad (14)$$

Here, $H = \dot{a}/a$ is the Hubble parameter and dots are differentiation with respect to t . Additionally, ρ_m and p_m are the energy density and pressure of the matter content. It is easy to prove that the Ricci scalar is $R = -T + B = 6(2H^2 + \dot{H})$, where the torsion scalar and the boundary term are $T = 6H^2$ and $B = 6(\dot{H} + 3H^2)$ respectively. Moreover, by setting $f = f(T)$ or $f = f(-T + B)$ in the above equations, we recover the standard $f(T)$ and $f(R)$ flat FLRW equations. Eqs. (13) and (14) can be also represented in a fluid form,

$$\kappa_{eff}^2(\rho_m + \rho_T) = 3H^2, \quad (15)$$

$$-\kappa_{eff}^2(\rho_m + p_m + \rho_T + p_T) = 2\dot{H}. \quad (16)$$

The above equations are analogous to standard FRW equations as in GR, the quantities appearing in these equations are defined in terms of $f(T, B)$ gravity as follows:

$$\frac{8\pi G}{3f_B + 2f_T} = \kappa_{eff}^2, \quad (17)$$

$$\frac{1}{\kappa^2} \left[\frac{T}{2}(3f_B + 2f_T) - 3H\dot{f}_B + 3\dot{H}f_B - \frac{1}{2}f(T, B) \right] = \rho_T, \quad (18)$$

$$\frac{1}{\kappa^2} \left[\frac{1}{2}f(T, B) - \frac{T}{2}(3f_B + 2f_T) - 3\dot{H}(3f_B + 2f_T) - 2H\dot{f}_T + \ddot{f}_B \right] = p_T. \quad (19)$$

It can be shown that the energy-momentum conservation equation holds for $f(T, B)$ gravity [10], therefore, we will have

$$\dot{\rho}_m + 3H(\rho_m + p_m) = 0. \quad (20)$$

Now, we have all the basis ingredients to study some properties in $f(T, B)$ cosmology as its thermodynamics and reconstructs some cosmological models.

III. THERMODYNAMICS OF $f(T, B)$ GRAVITY

Here, we intend to explore the validity of thermodynamic laws in generalized teleparallel theory of gravity in the non-equilibrium description. In the following section, we determine the restriction on parameters and model of $f(T, B)$ gravity for the validity of first and second laws of thermodynamics at the apparent horizon of FRW model. Also, we will show that for the total energy of the system to be positive, it is necessary that graviton is not a ghost in the sense of quantum gravity. We would like to mention that the results of non-equilibrium thermodynamics in $f(R)$ and $f(T)$ theories can be retrieved for some specific cases in this modified gravity.

A. First Law of Thermodynamics in $f(T, B)$ gravity

In order to discuss the thermodynamics of $f(T, B)$ gravity, we can find the dynamical apparent horizon by using the relation $h^{ab}\partial_a\tilde{r}\partial_b\tilde{r} = 0$. For the flat FLRW metric the radius of the apparent horizon is

$$\tilde{r}_A = \frac{1}{H}. \quad (21)$$

The time derivative of the above equation gives us

$$-\frac{d\tilde{r}_A}{\tilde{r}_A^3} = \dot{H}Hdt. \quad (22)$$

Using Eq. (16) in the above equation, one gets

$$\frac{(3f_B + 2f_T)d\tilde{r}_A}{4\pi G} = \tilde{r}_A^3 H(\rho_{eff} + p_{eff})dt. \quad (23)$$

Here, $\rho_{eff} = \rho_T + \rho_m$ and $p_{eff} = p_T + p_m$, are the total density and pressure of the universe.

Now we need to define the Bekenstein-Hawking horizon entropy in $f(T, B)$ gravity. For this purpose, we provide the review of such definition in GR as well as in some non-standard theories. In GR Bekenstein-Hawking horizon entropy is defined by $S_h = A/(4G)$, where $A = 4\pi\tilde{r}_A^2$ is the area of the apparent horizon [48]. In modified theories of gravity like $f(R)$ gravity, the horizon entropy S_h associated with the Noether charge, the so-called Wald entropy, can be defined by $S_h = A/(4G_{eff})$ [49], where $G_{eff} = G/f'(R)$ with $f'(R) = df(R)/dR$. We would like to mention that this definition of Wald entropy in $f(R)$ gravity is valid for both metric and Palatini formalism [50]. Similarly, in our notation, Wald entropy in $f(T)$ gravity is defined as $S_h = 2A/(4G_{eff})$, where $G_{eff} = G/f'(T)$ with $f'(T) = df(T)/dT$ [51]. Hence in newly proposed modified teleparallel

gravity theory, we define the Wald entropy as $S_h = A/(4G_{eff})$, where $G_{eff} = G/(3f_B + 2f_T)$. Hence, the Wald entropy in $f(T, B)$ reads as follows

$$S_h = \frac{A(3f_B + 2f_T)}{4G}. \quad (24)$$

Clearly, if we set $f = f(-T + B) = f(R)$ and $f = f(T)$ we recover the standard $f(R)$ and $f(T)$ Wald entropies relationships respectively. From Eqs. (23) and (24), we get

$$\frac{dS_h}{2\pi\tilde{r}_A} = \frac{\tilde{r}_A}{2G}d(3f_B + 2f_T) + 4\pi\tilde{r}_A^3 H(\rho_{eff} + p_{eff})dt. \quad (25)$$

The associated temperature of the apparent horizon is defined through the surface gravity κ_{sg} as

$$T_H = \frac{|\kappa_{sg}|}{2\pi}, \quad (26)$$

where κ_{sg} is given by [52]

$$\kappa_{sg} = \frac{1}{2\sqrt{-h}}\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta\tilde{r}_A) = -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right) = -\frac{\tilde{r}_A}{2}(2H^2 + \dot{H}). \quad (27)$$

By multiplying the term $\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)$ on both sides of Eq. (25), we get

$$T_H dS_h = 4\pi\tilde{r}_A^3 H(\rho_{eff} + p_{eff})dt - 2\pi\tilde{r}_A^2(\rho_{eff} + p_{eff})d\tilde{r}_A + \frac{\pi\tilde{r}_A^2 T_H}{G}d(3f_B + 2f_T). \quad (28)$$

In GR, the Misner-Sharp energy is defined as $E = \tilde{r}_A/(2G)$. In $f(T, B)$ gravity, this definition can be extended as

$$E = \frac{\tilde{r}_A(3f_B + 2f_T)}{2G}. \quad (29)$$

From Eqs.(21) and (29), we then get that the Misner-Sharp energy is

$$E = V \frac{3H^2(3f_B + 2f_T)}{4\pi G} = V\rho_{eff}, \quad (30)$$

where $V = (4/3)\pi\tilde{r}_A^3$, is the volume of the interior region of the apparent horizon. From the above equation, we find that E is the total intrinsic energy of the system. Also, we need to have $(3f_B + 2f_T) > 0$, so that $E > 0$. For this restriction on $(3f_B + 2f_T) > 0$, the effective gravitational coupling $G_{eff} = G/(3f_B + 2f_T)$ needs to be positive. We would like to mention that the condition $(3f_B + 2f_T) > 0$, is necessary condition to ensure that graviton is not a ghost in the sense of quantum gravity.

From Eqs.(15) and (29), one finds

$$dE = \frac{\tilde{r}_A}{2G}d(3f_B + 2f_T) + 4\pi\rho_{eff}\tilde{r}_A^2 d\tilde{r}_A - 4\pi H\tilde{r}_A^3(\rho_{eff} + p_{eff})dt. \quad (31)$$

By combining Eqs. (28) and (31), we obtain

$$T_H dS_h = -dE + 2\pi\tilde{r}_A^2(\rho_{eff} + p_{eff})d\tilde{r}_A + \frac{\tilde{r}_A}{2G}(2\pi\tilde{r}_A T_H + 1)d(3f_B + 2f_T). \quad (32)$$

By defining the work density, we get

$$W = -\frac{1}{2}\left(T^{(M)\alpha\beta}h_{\alpha\beta} + T^{(de)\alpha\beta}h_{\alpha\beta}\right) = \frac{1}{2}(\rho_{eff} + p_{eff}). \quad (33)$$

Here, $T^{(de)\alpha\beta}h_{\alpha\beta}$ is the energy-density of the dark components. Using the above definition of the work density in Eq.(32), we arrive at

$$T_H dS = -d\bar{E} + \bar{W}dV + \frac{\tilde{r}_A}{2G}(2\pi\tilde{r}_A T_H + 1)d(3f_B + 2f_T), \quad (34)$$

which can be re-written as

$$T_H dS_h + T_H d\bar{S} = -dE + WdV, \quad (35)$$

where $d\bar{S} = -(\tilde{r}_A/(2GT_H))(2\pi\tilde{r}_A T_H + 1)d(3f_B + 2f_T)$. The extra term $d\bar{S}$ defined in Eq. (35) can be treated as an entropy production term in non-equilibrium thermodynamics. In $f(T, B)$ gravity $d\bar{S} \neq 0$, due to $d(3f_B + 2f_T) \neq 0$. In GR and alternative theories including Gauss-Bonnet and Lovelock gravities [53], the usual FLT is satisfied by the respective field equations. In fact these theories do not involve any surplus term in universal form of FLT *i.e.*, $TdS = -dE + WdV$. The results in $f(R)$ and $f(T)$ theories can be retrieved for some specific cases in this modified gravity. Here, we may define the effective entropy term being the sum of horizon entropy and entropy production term as $S_{eff} = S_h + \bar{S}$ so that Eq.(35) can be rewritten as

$$T_h dS_{eff} = -dE + WdV, \quad (36)$$

where S_{eff} is the effective entropy related to the contributions from torsion scalar and boundary term at the apparent horizon of FLRW spacetime.

B. Second Law of Thermodynamics in $f(T, B)$ gravity

In order to investigate the second law of thermodynamics in $f(T, B)$ gravity, we start with the Gibbs equation in terms of matter and dark energy components, given by

$$T_t dS_t = d(\rho_{eff}V) + p_{eff}dV = Vd(\rho_{eff}) + (\rho_{eff} + p_{eff})dV, \quad (37)$$

where S_t denotes the total entropy of the system inside the horizon. It is natural to assume that the total temperature of energy source inside the horizon is proportional to the temperature of the

apparent horizon *i.e.*, $T_{tot} = bT_h$ where $0 < b < 1$. It may result in local equilibrium by setting the proportionality constant as unity as mentioned in [54]. Generically, the horizon temperature differ from the temperature of all energy sources inside the horizon and the systems must experience interaction for some interval of time ahead of attaining the thermal-equilibrium. Furthermore, mutual coupling of matter and curvature components in this theory may result in spontaneous energy flow between the horizon and matter contents.

The validity of the generalised second law of thermodynamics (GSLT) requires the condition

$$\Omega \equiv \frac{dS_h}{dt} + \frac{d(d\bar{S})}{dt} + \frac{dS_t}{dt} \geq 0. \quad (38)$$

Now using the FLRW equations together with Eqs.(35) and (38), we find the following condition for validity of GSLT

$$\frac{2\pi}{bGH^3(\dot{H} + 2H^2)} \{2(1-b)\dot{H}H^2(3f_B + 2f_T) + (1-b)H^3\partial_t(3f_B + 2f_T) + (2-b)\dot{H}^2(3f_B + 2f_T)\} \geq 0, \quad (39)$$

Here, (39) is satisfied if $(3f_B + 2f_T) > 0$, $\partial_t(3f_B + 2f_T) \geq 0$, $H > 0$ and $\dot{H} \geq 0$. If $b = 1$, *i.e.*, temperature between outside and inside the horizon remains the same then the GSLT is valid only if

$$\frac{2\pi(3f_B + 2f_T)\dot{H}^2}{G(\dot{H} + 2H^2)H^3} \geq 0. \quad (40)$$

Hence, we conclude that the GSLT can be met in phantom cosmic era.

IV. RECONSTRUCTION METHOD IN $f(T, B)$ COSMOLOGY

In this section, we will apply the usual reconstruction method to find specific form of the function $f(T, B)$ which mimics different cosmological models. Hereafter, we will assume that the matter pressure $p_m = w\rho_m$ where w is the state parameter. Therefore, by using the conservation equation (20), we find

$$\rho_m(t) = \rho_0 a(t)^{-3(w+1)}. \quad (41)$$

A. Power-law Cosmology

It would be interesting to explore the existence of exact power solutions in $f(T, B)$ gravity theory corresponding to different phases of cosmic evolution. Let us consider a model described by

a power-law scale factor given by

$$a(t) = \left(\frac{t}{t_0}\right)^h, \quad (42)$$

where t_0 is some fiducial time and h is a parameter greater than one. These solutions help to explain the cosmic history including matter/radiation and dark energy dominated eras. Further, these solutions provide the scale factor evolution for the standard fluids such as dust ($h = 2/3$) or radiations ($h = 1/2$) dominated eras of the Universe if GR is assumed as underlying valid gravitational field theory. Also, $h > 1$ predicts a late-time accelerating Universe. We would like to mention that h is arbitrary constant in the following form of power law solutions. For the above scale factor, the scalar torsion and boundary read as follows

$$T = \frac{6h^2}{t^2}, \quad (43)$$

$$B = \frac{6h(3h-1)}{t^2}. \quad (44)$$

Now, for simplicity, we will assume that the function can be written in the following form

$$f(T, B) = f_1(T) + f_2(B). \quad (45)$$

By inverting (43) and (44), the 00 equation given by (13) becomes

$$\frac{1}{2}f_1(T) - T f_1'(T) - \kappa^2 \rho_m(t) = K, \quad (46)$$

$$-2B^2 f_2''(B) + (1-3h)B f_2'(B) + (3h-1)f_2(B) = (2-6h)K. \quad (47)$$

Here, K is a constant for the method of separation variable. We can directly solve the above equations to obtain

$$f_1(T) = \frac{2\kappa^2 \rho_0}{1-3h(w+1)} \left(\frac{t_0}{\sqrt{6}h} \sqrt{T}\right)^{3h(w+1)} + C_1 \sqrt{T} + 2K, \quad (48)$$

$$f_2(B) = C_2 B^{\frac{1}{2}(1-3h)} + C_3 B - 2K. \quad (49)$$

Note that this is one specific form of the function which mimics a power-law cosmology. There are other possible functions that also will represent this model.

B. de-Sitter reconstruction

If we assume that the universe is governed by a de-Sitter form, i.e., the scale factor of the universe is an exponential $a(t) \propto e^{H_0 t}$, both the torsion scalar and the boundary term are constants.

Explicitly they are given by $T = 6H_0^2$ and $B = 18H_0^2$ respectively. This kind of evolution of the universe is very well known and important since it correctly describes the expansion of the current universe. For this kind of universe, it is known that the universe must be filled by a dark energy fluid whose state parameter $w = -1$ and hence the energy density is also a constant. From Eq. (13), it is easily to see that any kind of functions of $f(T, B)$ can admit de-Sitter solution if the following constraint is satisfied,

$$H_0^2 (9f_B(T_0, B_0) + 6f_T(T_0, B_0)) - \frac{1}{2}f(T_0, B_0) + \kappa^2\rho_0 = 0. \quad (50)$$

For instance, by assuming that the function is separable as $f(T, B) = f_1(B) + f_2(T)$, a possible reconstruction function which describes a de-Sitter universe is given by

$$f(T, B) = 2\kappa^2\rho_0 + f_0 e^{\frac{B}{18H_0^2}} + \tilde{f}_0 e^{\frac{T}{12H_0^2}}, \quad (51)$$

which of course is a constant function. Here, f_0 and \tilde{f}_0 are integration constants.

C. Λ -CMD reconstruction

Here, we are interested to discuss the reconstruction of the $f(T, B)$ function for a Λ -CDM cosmological evolution in the absence of any cosmological constant term in the modified Einstein field equations. This model was firstly formulated by Elizalde et al. [56] in $f(R, G)$ modified theory of gravity. The cosmological effects of the cosmological constant term in the concordance model is exactly replaced by the modification introduced by $f(T, B)$ function with respect to the usual Einstein-Hilbert Lagrangian.

For simplicity, instead of working with all the variables depending on the cosmic time t , we will use the e-folding parameter defined as $N = \ln(a/a_0) = -\ln(1+z)$. In terms of this variable, we can express $a(t)$, $H(t)$ and time derivatives as

$$a = a_0 e^N, \quad H = \frac{\dot{a}}{a} = \frac{dN}{dt}, \quad \frac{d}{dt} = H \frac{d}{dN}.$$

Therefore, we can rewrite equation (13) in terms of this quantity, yielding

$$-3H^2(3f_B + 2f_T) + 18H \left[(H^2 H'' + H H'^2 + 6H^2 H') f_{BB} + 2H^2 H' f_{BT} \right] - 3H H' f_B + \frac{1}{2} f(T, B) = \kappa^2 \rho(t). \quad (52)$$

Additionally, in term of the e-folding, the scalar torsion and the boundary term are $T = 6H^2$ and $B = 6H(3H + H')$ respectively. Now, for convenience, we introduce a new variable $g = H^2$ making

that the above equation becomes

$$-\frac{3}{2}(g' + 6g)f_B + 18gg'f_{TB} + 9gf_{BB}(g'' + 6g') - 6gf_T + \frac{1}{2}f(T, B) = \kappa^2\rho_m. \quad (53)$$

It is easily to compute that the torsion scalar and the boundary term written in this variable are $T = 6g$ and $B = 3(g' + 6g)$ respectively. Now, we will assume that the function $f(T, B)$ is separable as Eq. (45). Under these assumptions, the above equation becomes

$$-\frac{3}{2}(g' + 6g)f_1'(B) + 9gf_1''(B)(g'' + 6g') + \frac{1}{2}f_1(B) = \kappa^2\rho_m - \frac{1}{2}f_2(T) + 6gf_2'(T), \quad (54)$$

Let us now study a Λ -CMD model whose function $g = g(N)$ is given by

$$g = H_0^2 + le^{-3N}, \quad l = \frac{\kappa^2\rho_0 a_0^{-3}}{3}. \quad (55)$$

Under this model, the e-folding can be expressed depending on the boundary term as follows

$$N = \frac{1}{3} \log \left(\frac{9l}{B - 18H_0^2} \right). \quad (56)$$

Therefore, we can rewrite Eq. (54) as follows

$$\frac{1}{6} (4(B^2 - 27BH_0^2 + 162H_0^4)f_1''(B) - 3Bf_1'(B) + 3f_1(B)) = \frac{K}{2}, \quad (57)$$

$$\kappa^2\rho_0 \left(\frac{6H_0^2 - T}{-6la_0^3} \right)^{w+1} - \frac{1}{2}f_2(T) + Tf_2'(T) = -\frac{K}{2}, \quad (58)$$

where K is a constant since the r.h.s. of (54) depends only on T and the l.h.s. only on B . Note that the energy density can be expressed depending on T or B so that, the above equations have one of the possible options to reconstruct a Λ -CMD universe. Thus, one possible way to reconstruct this model is by taking the following functions

$$f_1(B) = \frac{2C_2 \sqrt[4]{B - 9H_0^2} \left(B \sqrt{18 - \frac{B}{H_0^2}} {}_2F_1 \left(\frac{1}{4}, \frac{1}{2}; \frac{5}{4}; \frac{B}{9H_0^2} - 1 \right) - 3B + 54H_0^2 \right)}{3\sqrt{B - 18H_0^2}} + BC_1 + K, \quad (59)$$

$$f_2(T) = \left(6 - \frac{T}{H_0^2} \right)^{-w} \left[3a_0^3 l \left(C_3 \sqrt{T} + K \right) \left(6 - \frac{T}{H_0^2} \right)^w - \kappa^2\rho_0 T {}_2F_1 \left(\frac{1}{2}, -w; \frac{3}{2}; \frac{T}{6H_0^2} \right) \left(\frac{T - 6H_0^2}{a_0^3 l} \right)^w - 6H_0^2 \kappa^2\rho_0 {}_2F_1 \left(-\frac{1}{2}, -w; \frac{1}{2}; \frac{T}{6H_0^2} \right) \left(\frac{T - 6H_0^2}{a_0^3 l} \right)^w \right]. \quad (60)$$

Here, C_1 , C_2 and C_3 are constants and ${}_2F_1$ represents the hypergeometric function of the second kind.

D. Phantom behaviour

As an another example, let us now assume that the Hubble parameter and the energy density are

$$\sqrt{g} = H = e^{mN}, \quad \rho_m = b_1 + \frac{3}{\kappa^2} e^{2mN} + \frac{96(m+1)}{5} b_2 e^{5mN}, \quad (61)$$

where b_1 , b_2 and m are constants. Cosmologically speaking, this model represents a super accelerated universe phase with a phantom regime $w_{eff} < -1$, making that the universe could end in a singularity. By using Eq. (54) we again can split this equation depending only on T and B as follows

$$\frac{B(Bm(2m+1)f_1''(B) - (m+3)^2 f_1'(B)) + (m+3)^2 f_1(B)}{2(m+3)^2} = \frac{K}{2}, \quad (62)$$

$$\kappa^2 b_1 + \frac{4}{15} \sqrt{\frac{2}{3}} \kappa^2 b_2 (m+1) T^{5/2} + \frac{T}{2} + T f_2'(T) - \frac{f_2(T)}{2} = -\frac{K}{2}, \quad (63)$$

where K is a constant. Thus, one of the possible representation which produces a super accelerated universe are given by the following functions,

$$f_1(B) = K + C_1 B^{\frac{(m+3)^2}{m(2m+1)}} + C_2 B, \quad (64)$$

$$f_2(T) = K - T + 2\kappa^2 b_1 - \frac{2}{15} \sqrt{\frac{2}{3}} \kappa^2 b_2 (m+1) T^{5/2} + C_3 \sqrt{T}, \quad (65)$$

where C_1 , C_2 and C_3 are integration constants.

E. Reconstruction method in $f(T, B) = -T + F(B)$ cosmology

In this section we will study the specific case where the function takes the form $f(T, B) = -T + F(B)$, which is similar to models of the form $f(R) = R + F(R)$ and $f(T) = -T + f(T)$ studied in $f(R)$ and $f(T)$ gravity respectively [55]. This theory is equivalent than to consider a teleparallel background (or GR) plus an additional function which depends on the boundary term which can be also understood as $F(B) = F(T + R)$. In this model, the 00 field equation (13) becomes

$$-3H^2(3F_B - 2) + 3H\dot{F}_B - 3\dot{H}F_B + \frac{1}{2}(F(B) - 6H^2) = \kappa^2 \rho_m, \quad (66)$$

where the energy density is given by (41). Equivalently, from (52), it is easily to rewrite the above equation in term of the e-folding,

$$-3H^2(3F_B - 2) + 18H \left[(H^2 H'' + H H'^2 + 6H^2 H') F_{BB} \right] - 3H H' F_B + \frac{1}{2}(-6H^2 + F(B)) = \kappa^2 \rho_m. \quad (67)$$

Let us now perform a reconstruction method for all the same models as we did in the previous section.

For a power-law cosmology described in Sec. IV A, Eq. (66) can be written as follows,

$$\frac{B^2 F'(B)}{1-3h} + \frac{BF(B)}{6h-2} + \frac{Bh(3F(B)-2)}{2-6h} + \frac{1}{2} \left(F(B) + \frac{Bh}{1-3h} \right) = \kappa^2 \rho_0 \left(6^{h/2} \left(\frac{\sqrt{h(3h-1)}}{\sqrt{B}t_0} \right)^h \right)^{-3(w+1)}, \quad (68)$$

which can be directly solved if we assume the vacuum case ($\rho_0 = 0$) obtaining

$$F(B) = \frac{2^{-\frac{3h}{2}-1} B^{\frac{1}{2}-\frac{3h}{2}} e^{\frac{1-3h}{2B}}}{\sqrt{1-3h}} \left[\sqrt{2}h(1-3h)^{\frac{3h}{2}} \Gamma \left(\frac{1}{2} - \frac{3h}{2}, \frac{1-3h}{2B} \right) + 2^{\frac{3h}{2}+1} C_1 \sqrt{1-3h} - \sqrt{2}h(1-3h)^{\frac{3h}{2}} \Gamma \left(\frac{1}{2} - \frac{3h}{2}, \frac{1}{2} - \frac{3h}{2} \right) \right]. \quad (69)$$

where C_1 is an integration constant and Γ represents the Euler Gamma function. For simplicity we have assumed the vacuum case, otherwise the complete expression will depend on integrals.

Now, for a de-Sitter reconstruction, the scale factor behaves as $a(t) = a_0 e^{H_0 t}$, then $B = 18H_0^2$ and hence from (66) we directly find that the function takes the following form,

$$F(B) = \frac{B - 6\kappa^2 \rho_0}{3(B-1)}. \quad (70)$$

Let us now reconstruct a Λ -CMD universe where $g = H_0^2 + l e^{-3N}$. In this theory, Eq. (54) becomes

$$-(B - 18H_0^2)(B - 9H_0^2) F''(B) - \frac{1}{2} B F'(B) + \frac{F(B)}{2} + \frac{1}{3} (B - 9H_0^2) = 3^{-2(w+1)} \kappa^2 \rho_0 \left(a_0^3 \sqrt[3]{\frac{l}{B - 18H_0^2}} \right)^{-3(w+1)}, \quad (71)$$

where we have used Eq. (56) to express all in term of the boundary term B . By assuming $\rho_0 = 0$, we find that the function is

$$F(B) = \frac{C_2 \left(3H_0 \sqrt{B - 9H_0^2} - B \tanh^{-1} \left(\frac{\sqrt{B - 9H_0^2}}{3H_0} \right) \right)}{54H_0^3} + BC_1 + \frac{1}{18} (4B \log(B - 18H_0^2) + B - 36H_0^2), \quad (72)$$

where C_1 and C_2 are integration constants.

V. PERTURBATIONS AND STABILITY

In this section, we are interested to establish the stability conditions for cosmological solutions against linear isotropic homogeneous perturbations in $f(T, B)$ theory of gravity. We formulate the

perturbation equations in FLRW universe for both general as well as specific cases in particular, the stability of de-Sitter and power law solutions. We assume a general solution

$$H(t) = H_j(t), \quad (73)$$

which satisfies the basic equations of motion of FLRW universe in $f(T, B)$ theory of gravity. In term of above solution, the torsion scalar and boundary B , can be written as follows

$$T = 6\dot{H}_j^2(t), \quad (74)$$

$$B = 6\dot{H}_j(t) - 18H_j^2(t). \quad (75)$$

If we consider particular model of $f(T, B)$, that can generate solution (73), then following equations must be satisfied

$$-3H_j^2(3f_B^j + 2f_T^j) + 3H_j\dot{f}_B^j - 3\dot{H}_j f_B^j + \frac{1}{2}f^j = \kappa^2\rho_j, \quad (76)$$

$$\dot{\rho}_j + 3H_j(1 + \omega)\rho_j = 0. \quad (77)$$

Now we define the perturbation for Hubble parameter and energy density as follows

$$H(t) = H_j(t)(1 + \delta(t)), \quad \rho(t) = \rho_j(1 + \delta_m(t)). \quad (78)$$

Here, our purpose is to make the perturbation analysis about the solution $H(t) = H_j(t)$, so that function $f(T, B)$ can be expressed in the powers of T and B as

$$f(T, B) = f^j + f_T^j(T - T_j) + f_B^j(B - B_j) + \mathcal{O}^2, \quad (79)$$

here, the superscript j means the values of $f(T, B)$ and its derivatives are evaluated at $T = T_0$ and $B = B_0$. The term \mathcal{O}^2 includes all the terms which have power square and higher powers of T and B , although we shall only consider the linear terms of the defined perturbation. Thus, by replacing Eqs. (78) and (79) in the FLRW equation and in the continuity equation, we get the perturbation equations in terms of $\delta(t)$ and $\delta_m(t)$, (in the linear approximation) in the form of the following differential equations

$$c_2\ddot{\delta}(t) + c_1\dot{\delta}(t) + c_0\delta(t) = c_m\delta_m(t), \quad (80)$$

$$\dot{\delta}_m(t) + 3H_h\delta(t) = 0. \quad (81)$$

The coefficients $c_{0,1,2,m}$, are expressed in the Appendix. These depend explicitly on $f(T, B)$ and its derivatives evaluated at background solutions $H = H_j$. In general it is not easy to solve the above equations analytically. In the coming sections we shall present some particular models for the solution of above equations.

A. Stability of de-Sitter Solution

Consider the de-Sitter solution with $H_j = H_0$ and $\rho_0 = 0$, then the perturbed equation takes the following form,

$$\begin{aligned} & \left(-18H_0^2 f_{TB}^0 T_0 + 324H_0^4 f_{BB}^0 - 36H_0^2 f_B^0 - 12H_0^2 f_{TT}^0 T_0 + 216H_0^4 f_{TB}^0 \right. \\ & \left. - 24H_0^2 f_T^0 - 24H_0^2 f_B \right) \delta(t) + \left(-54H_0^3 f_{BB}^0 - 6H_0 f_B^0 \right) \dot{\delta}(t) + (-18H_0^2 f_{BB}^0) \ddot{\delta}(t) = 0. \end{aligned} \quad (82)$$

Using the $f(T, B)$ model Eq. (51) formulated in de-Sitter reconstruction, we get the solution for $\delta(t)$ as follows

$$\delta(t) = C_1 e^{\mu_+ t} + C_2 e^{\mu_- t}, \quad (83)$$

where C_1 and C_2 are integration constants and

$$\mu_{\pm} = \frac{3H_0}{2f_0} \left(-3f_0 \pm \sqrt{f_0 (f_0 - 28\sqrt{e}\tilde{f}_0)} \right). \quad (84)$$

Here, f_0 and \tilde{f}_0 are the constants appearing in Eq. (50). Note that $\sqrt{e} = e^{1/2}$ is referring to the exponential e and not the determinant of the tetrad. The growth of the perturbation will depend both upon the overall sign of the parameters μ_{\pm} appearing in the expression (84) and also upon the real and imaginary character of the square root. Thus four different cases can be distinguished:

- $f_0 < 0$ and $f_0 > 28\sqrt{e}\tilde{f}_0$ with $\tilde{f}_0 < 0$, this implies that solutions are complex and $\Re(\mu_{\pm}) < 0$, thus solutions behave as a damped oscillator of decreasing amplitude. Hence, solutions are stable.
- $f_0 > 0$ and $f_0 < 28\sqrt{e}\tilde{f}_0$ with $\tilde{f}_0 > 0$, this implies that solutions are complex and $\Re(\mu_{\pm}) < 0$, thus solutions behave as a damped oscillator of decreasing amplitude. Hence, solutions are stable.
- $0 < 28\sqrt{e}\tilde{f}_0/f_0 < 1$ with $f_0 > 0$ and $\tilde{f}_0 > 0$, or $f_0 < 0$ and $\tilde{f}_0 < 0$, then both μ_{\pm} are real and $\mu_{\pm} < 0$, hence solutions are stable.

B. Stability of Power Law Solutions

Now, we will study the stability of power law solutions of the form (42). For $0 < h < 1$, we have decelerated universe which may refer to dust dominated ($h = 2/3$) or radiation dominated

($h = 1/2$), while $h > 1$ results in accelerating picture of the universe. Here, we explore the stability of power law solutions for matter dominated, radiation dominated and late time accelerated eras.

- For matter dominated era with $h = 2/3$, and $\omega = 0$, equations (48) and (49) result in

$$f(T, B) = \frac{C_2}{\sqrt{B}} + C_3 B + C_1 \sqrt{T} - \frac{3}{4} \rho_0 \kappa^2 T. \quad (85)$$

By substituting the above model in equations (80) and (81), one can find the required perturbation equations for matter dominated power law model. Here, we employ the numerical approach to solve these equations and present the evolution of perturbation parameters $\delta(t)$ and $\delta_m(t)$. In this study we set $H_0 = 67$, $\Omega_m = 0.23$, $C_2 = -0.2$, $C_1 = C_3 = 0.1$ and $\kappa^2 = 1$. Figure 1 shows the oscillating behavior of $\delta(t)$ and $\delta_m(t)$, however these do not decay in future evolution.

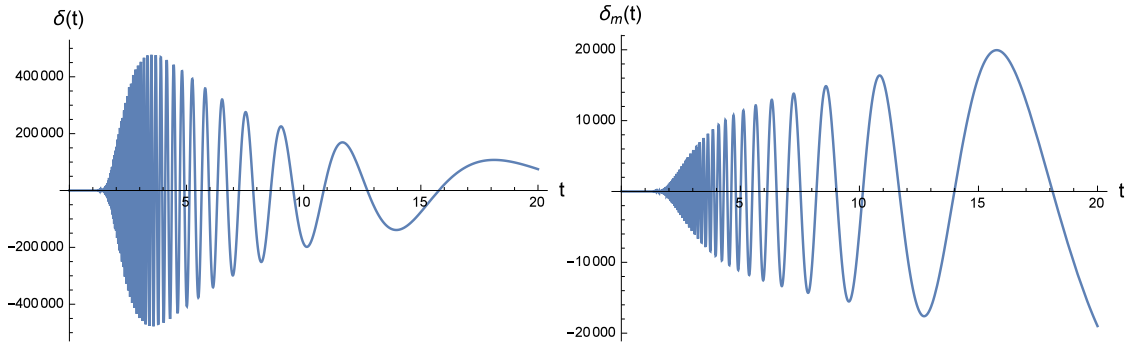


FIG. 1: Evolution of $\delta(t)$ and $\delta_m(t)$ versus time t . Herein, we set the initial conditions $\delta'(1) = 0.2$, $\delta(1) = 0.1$ and $\delta_m(1) = 0.1$.

- For radiation dominated era with $h = 1/2$, and $\omega = 1/3$, equations (48) and (49) result in

$$f(T, B) = \frac{C_2}{B^{1/4}} + C_3 B + C_1 \sqrt{T} - \frac{4}{3} \kappa^2 \rho_0 T. \quad (86)$$

One can substitute the model (86) in equations (80) and (81) to find the required perturbation equations for radiation dominated power law model. We use the numerical scheme and show the evolution of $\delta(t)$ and $\delta_m(t)$ in Figure 2. This figure also shows the oscillating behavior of $\delta(t)$ and $\delta_m(t)$, however the oscillations of $\delta(t)$ decays in future while that of $\delta(t)_m$ does not decay in future. Hence solutions are unstable as full perturbation around a cosmological solution is fully determined by the matter perturbations. This result is similar to matter dominated era with $h = 2/3$, and $\omega = 0$ which is shown in figure 1.

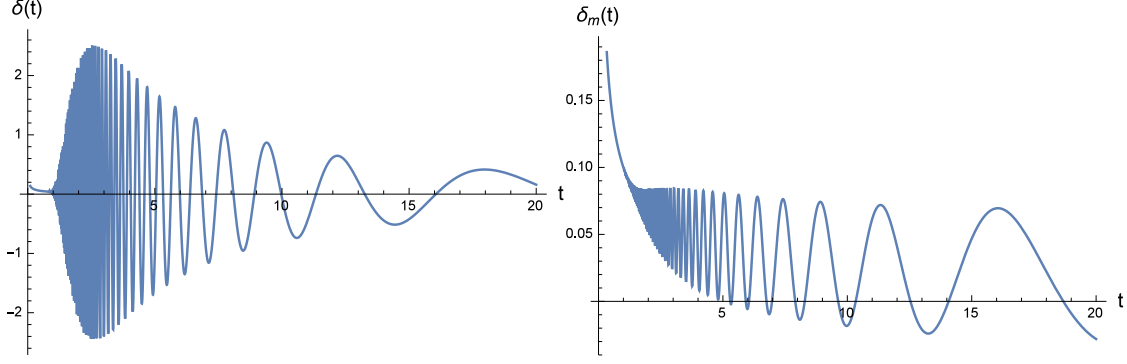


FIG. 2: Evolution of $\delta(t)$ and $\delta_m(t)$ versus time t . Herein, we set the initial conditions $\delta'(1) = 0.2$, $\delta(1) = 0.1$ and $\delta_m(1) = 0.1$.

- For the choice of $h > 1$, we have expanding behavior of the cosmos. In our case we set $h = 2$ with $\omega = -0.5$, so that the corresponding power law model is given by

$$f(T, B) = \frac{C_2}{B^{\frac{5}{2}}} + C_3 B + C_1 \sqrt{T} - \frac{\kappa^2 \rho_0 T^{\frac{3}{2}}}{48\sqrt{6}}. \quad (87)$$

Again following a similar approach, we show the results in Figure 3. Here, we set $C_1 = 0.1$ and $C_2 = -10$, $C_3 = -1$. For this case it can be seen that $\delta(t)$ and $\delta_m(t)$ decay in later times so that power law model (87) is stable.

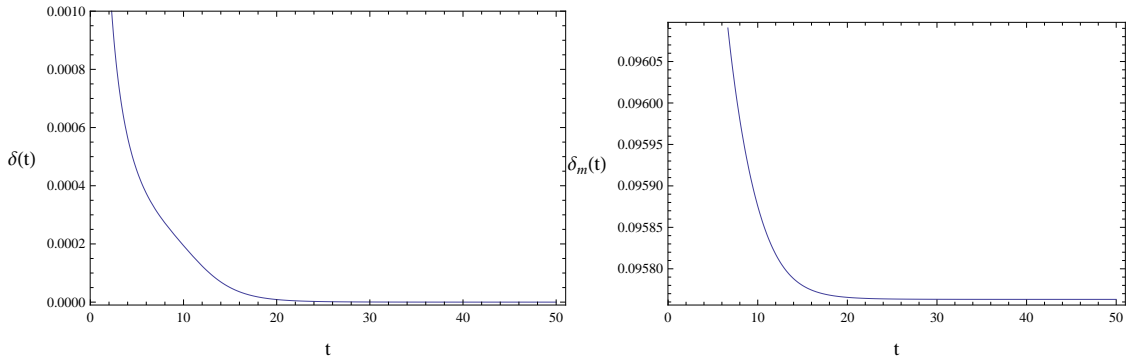


FIG. 3: Evolution of $\delta(t)$ and $\delta_m(t)$ versus time t . Herein, we set the initial conditions $\delta'(1) = 0.2$, $\delta(1) = 0.1$ and $\delta_m(1) = 0.1$.

VI. CONCLUSIONS

Over the last years, teleparallel theories of gravity and its modifications have been very studied in cosmology. These theories lie in a globally flat manifold endorsed with torsion. It is well-known

that GR has an equivalent teleparallel representation (TEGR) based on the torsion (and tetrads) instead of curvature (and metric). In order to describe the dark energy behaviour, different modified teleparallel theories have been proposed. The first one, the so-called, $f(T)$ gravity is a natural generalisation of the TEGR action by changing the torsion scalar $T \rightarrow f(T)$ in the action. This approach is analogous with $f(R)$ gravity in the metric counterpart. These two theories have been very successful describing the cosmological behaviour of the universe. With the aim to unify both $f(R)$ and $f(T)$ gravity and see how these theories are connected, it was formulated a modified teleparallel theory of gravity called $f(T, B)$ which under suitable limits can become $f(T)$ or $f(R)$ gravity [10]. In general, this theory is a fourth-order one and it is not local under Lorentz transformations. In this work, we have explored different cosmological features in $f(T, B)$ gravity as its thermodynamics and its possible use to reconstruct feasible cosmological models.

We have shown that the modified flat FLRW equations under this theory can be cast to the form of first law of thermodynamics, $T_h dS_h + T_h d\bar{S} = -dE + W dV$ in a non-equilibrium description of thermodynamics. In this structure of first law, we have found that entropy S_{eff} involves contribution from two factors, the first corresponds to horizon entropy in terms of area and second represents the entropy production term $d\bar{S}$ which is produced due to the non-equilibrium description in $f(T, B)$ gravity. This shows that one may need the non-equilibrium treatment of thermodynamics in this modified theory. The entropy production term in $f(T, B)$ gravity is more general and can reproduce the corresponding factor in $f(-T + B) = f(R)$ and $f(T)$ theories. It is worth mentioning that no such term is present in Einstein, Gauss-Bonnet, Lovelock and braneworld modified theories [53]. We have also explored the validity of GSLT at the apparent horizon of FLRW universe in this modified theory. The time evolution of entropy S_{tot} summed up with the contributions from horizon entropy and entropy associated with the matter contents within the horizon is presented in a comprehensive way, where we have assumed the proportionality relation between the temperatures related to apparent horizon and matter components inside the horizon. The condition $T_{in} = bT_h$ relates the temperature of ingredients inside the horizon to the temperature of apparent horizon. As from macroscopic viewpoint, temperature is considered as the property shared by two systems initially at different states and then have been placed in thermal contact would result in thermal equilibrium. In case of thermal equilibrium, all the fluids in cosmos assume the same temperature which is more or less equal to the horizon temperature. This situation will certainly be the case of late times when cosmic components would have interacted for long time.

Cosmological reconstruction of $f(T, B)$ gravity was also presented in the background of power law and de Sitter evolution. These solutions explain the matter/radiation dominated phase that

connects with the accelerating epoch. Moreover, we have used a more comprehensive approach in terms of e-folding representing different eras of the universe to formulate the Λ CDM model and super accelerated universe model. Hence, $f(T, B)$ gravity can reproduce the Λ CDM expansion and transition from deceleration to accelerated expansion without any additional constraint. Additionally, we have explored the specific case $f(T, B) = -T + F(B)$, which is a GR background plus an additional function which depends on the boundary term. The reconstruction scheme is carried out for this specific case and we have also found the corresponding function $F(B)$ which mimics different cosmological models.

The study of stability/instability of various forms of Lagrangian is a useful tool to classify the modified theories on physical grounds. In this work, we have explored the stability of $f(T, B)$ models which reproduces the de-Sitter and power law expansion history. We have derived the complete set of linear perturbation equations in this modified theory and presented the coefficients in explicit form. We set the stability conditions for de-Sitter model as presented in Sec. IV.B. Moreover, in case of power law model, the perturbations have been discussed for three cases namely matter dominated, radiation dominated and accelerated expansion. It can be seen the accelerated expansion model is stable.

Appendix A

$$\begin{aligned}
c_0 = & \left(-18H^2 {}_h f_{TB}^j T_j + 324H_j^4 f_{BB}^j + 54H_j^2 \dot{H}_j f_{BB}^j - 36H_j^2 f_B^j - 12H_j^2 f_{TT}^j T_j \right. \\
& + 216H_j^4 f_{TB}^j + 36H_j^2 \dot{H}_j f_{TB}^j - 24H_j^2 f_T^j + 6H_j \dot{f}_{TB}^j T_j + 6H_j f_{TB}^j \dot{T}_j - 108H_j^3 \dot{f}_{BB}^j \\
& - 18\dot{H}_j H_j \dot{f}_{BB}^j - 216H_j^2 \dot{H}_j f_{BB}^j - 18H_j \ddot{H}_j f_{BB}^j + 6H_j \dot{f}_B^j - 6\dot{H}_j T_j f_{TB}^j \\
& \left. + 108\dot{H}_j H_j^2 f_{BB}^j + 18\dot{H}_j^2 f_{BB}^j - 9\dot{H}_j f_B^j + f_B^j T_j - 18f_B^j H^2 - 3f_B^j \dot{H}_j \right) \quad (A.1)
\end{aligned}$$

$$\begin{aligned}
c_1 = & 54H_j^3 f_{BB}^j + 36H_j^3 f_{TB}^j + 6H_j f_{TB}^j T_j - 18H_j^2 \dot{f}_{BB}^j - 108H_j^3 f_{BB}^j \\
& - 6H_j f_B^j \quad (A.2)
\end{aligned}$$

$$c_2 = -18H^2 f_B^j B \quad (A.3)$$

$$c_m = \kappa^2 \rho_m \quad (A.4)$$

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